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Explicitly quasiconvex set-valued optimization

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Abstract The principal aim of this paper is to extend some recent results concerning the contractibility of efficient sets and the Pareto reducibility in multicriteria explicitly quasiconvex optimization problems to similar vector optimization problems involving set-valued objective maps. To this end, an appropriate notion of generalized convexity is introduced for set-valued maps taking values in a partially ordered real linear space, which naturally extends the classical concept of explicit quasiconvexity of real-valued functions. Actually, the class of so-called explicitly cone-quasiconvex set-valued maps in particular contains the cone-convex set-valued maps, and it is contained in the class of cone-quasiconvex set-valued maps.

Keywords Explicitly cone-quasiconvex functions · Set-valued optimization · Contractibility · Pareto reducibility

Mathematics Subject Classification (2000): 90C29 · 26E25 · 26B25

1 Introduction

Multiple criteria optimization problems involving explicitly quasiconvex (i.e., both quasiconvex and semistrictly quasiconvex) objective functions have been intensively studied in the literature, especially because these functions enjoy certain properties of convex functions, which allowed the authors to extend many valuable results from convex optimization to nonconvex (in particular fractional) optimization.

This work is motivated by two recent papers of Benoist (2003) and Popovici (2005). The first one concerns the contractibility of the efficient outcome set in explicitly quasiconvex multicriteria optimization, continuing a series of papers on the connectedness

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and contractibility of efficient sets, initiated by the early work of Schaible (1985), and followed among others by Sun (1996), Daniilidis et al. (1997), Benoist (1998), and Benoist and Popovici (2000). The second one deals with Pareto reducibility of multicriteria problems (i.e., the representation of the weakly efficient solution set as the union of the sets of efficient solutions of all subproblems obtained from the original one by selecting certain criteria), extending some previous results of Lowe et al. (1984) and Malivert and Boissard (1994). The junction point between these two distinct research directions is the concept of simply shaded set, previously introduced by Benoist and Popovici (2000), which is shown to be intimately related to explicit quasiconvexity.

The principal aim of this paper is to extend the aforementioned results from multicriteria explicitly quasiconvex optimization problems to similar optimization problems involving set-valued objective maps. To this end, an appropriate notion of generalized convexity is introduced for set-valued maps, which naturally extends the concept of explicit quasiconvexity of real-valued functions.

We begin in Sect. 2 by presenting some general definitions and preliminary results concerning cone-convex and cone-quasiconvex set-valued maps with values in a real linear space, partially ordered by a relatively solid convex cone. In Sect. 3, we introduce the notion of explicitly cone-quasiconvex set-valued map. Characterizations of this notions and the relationship between this notion and the usual notions of cone-convexity and cone-quasiconvexity are studied by keeping the same general framework as in Sect. 2. Then, in Sect. 4, by restricting our attention on set-valued maps with values in a finite dimensional Euclidean space, partially ordered by the standard ordering cone, we derive from our main result (Theorem 4.1) sufficient conditions for the Pareto reducibility, as well as for the contractibility of the efficient outcome set.

2 Cone-convexity and cone-quasiconvexity

Throughout this paper S will denote a nonempty convex subset of a real linear space X. Recall that a real-valued function $f : S \to \mathbb{R}$ is called:

- convex, if $f((1-t)x + tx') \leq (1-t)f(x) + tf(x')$ for all $x, x' \in S$ and $t \in [0, 1]$;
- quasiconvex, if $f((1-t)x + tx') \leq \max\{f(x), f(x')\}$ for all $x, x' \in S$ and $t \in [0, 1]$;
- semistrictly quasiconvex, if $f((1-t)x + tx') < \max\{f(x), f(x')\}$ for all $x, x' \in S$ with $f(x) \neq f(x')$ and $t \in]0, 1[$;
- explicitly quasiconvex, if it is both quasiconvex and semistrictly quasiconvex.

For classical multicriteria optimization purposes, these notions have been naturally adapted to functions with values in a finite-dimensional Euclidean space, as follows: a function $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ ($n \in \mathbb{N}$, $n \geq 2$) is called componentwise convex (resp., quasiconvex; semistrictly quasiconvex; explicitly quasiconvex) if each of the component functions $f_1, \ldots, f_n$ is convex (resp., quasiconvex; semistrictly quasiconvex; explicitly quasiconvex). However, in general vector optimization, where the image space of the objective function is an arbitrary partially ordered space and no componentwise approach makes sense, the classical notions of convexity and quasiconvexity have been extended for vector-valued functions and also for set-valued maps in various ways. The reader can find some of them in the works of Borwein (1977), Jahn (1986), Jeyakumar et al. (1993), Kuroiwa (1996), and Luc (1990). The aim of this Section is to study two of these notions. We start by recalling some preliminary notions and results concerning the convex cones.

Let Y be a linear space over the field $\mathbb{R}$ of real numbers. As usual, given a subset A of Y, we will denote by

$$\begin{aligned}\operatorname{cor} A &:= \{x \in A \mid \forall\, y \in Y, \exists\, \lambda \in \mathbb{R}^*_+, x + [0,\lambda]\cdot y \subset A\},\\ \operatorname{icr} A &:= \{x \in A \mid \forall\, y \in \operatorname{span}(A - A), \exists\, \lambda \in \mathbb{R}^*_+, x + [0,\lambda]\cdot y \subset A\}\end{aligned}$$

the algebraic interior and the relative algebraic interior of A, respectively. The set A is called solid (resp., relatively solid) if $\operatorname{cor} A \neq \emptyset$ (resp. $\operatorname{icr} A \neq \emptyset$). Recall also that A is said to be vectorially closed (in sense of Adán and Novo, 2003) if $A = \operatorname{vcl} A$, where the so-called vectorial closure of A is given by

$$\begin{aligned}\operatorname{vcl} A &:= \{y \in Y \mid \exists\, y' \in Y, \forall\, \lambda \in \mathbb{R}^*_+, \exists\, \lambda' \in]0,\lambda],\ y + \lambda' y' \in A\}\\ &= \{y \in Y \mid \exists\, y' \in Y, \exists\, \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^*_+,\ \lambda_n \to 0,\ y + \lambda_n y' \in A, \forall n \in \mathbb{N}\}.\end{aligned}$$

Obviously, for any $A \subset Y$ we have $\operatorname{icr} A \subset A \subset \operatorname{vcl} A$. By Proposition 3 in the paper of Adán and Novo (2003) it follows that if A is convex then

$$\forall\, t \in [0,1[\ :\ (1-t)\cdot \operatorname{icr} A + t\cdot \operatorname{vcl} A \subset \operatorname{icr} A. \tag{1}$$

In the sequel we will assume that the real linear space Y is partially ordered by a relatively solid convex cone C, i.e., $\emptyset \neq \operatorname{icr} C \subset C = \mathbb{R}_+\cdot C = C + C \subset Y$. Actually we endow the space Y with two binary relations, defined for any $y, y' \in Y$ by

$$y \leq_C y' \ :\Longleftrightarrow\ y' \in y + C \quad \text{and} \quad y <_C y' \ :\Longleftrightarrow\ y' \in y + \operatorname{icr} C.$$

In the particular case where $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$ ($n \in \mathbb{N}$), the binary relations $\leq_C$ and $<_C$ will be simply denoted by $\leq$ and $<$ (i.e., the usual component-wise order relations).

Taking into account that C is a relatively solid convex cone and observing that

$$\mathbb{R}^*_+\cdot \operatorname{icr} C = \operatorname{icr} C \quad \text{and} \quad \mathbb{R}_+\cdot \operatorname{vcl} C = \operatorname{vcl} C \tag{2}$$

it can be easily deduced from (1) that

$$\operatorname{icr} C + \operatorname{vcl} C = \operatorname{icr} C. \tag{3}$$

The following preliminary result gives some useful representations of the relative interior and the vectorial closure of the ordering cone.

Lemma 2.1 *For every $e \in \operatorname{icr} C$ we have*

$$\operatorname{icr} C = \bigcup_{\alpha>0} (\alpha e + C) \quad \textit{and} \quad \operatorname{vcl} C = \bigcap_{\beta<0} (\beta e + \operatorname{icr} C).$$

Proof Let $e \in \operatorname{icr} C$. Then, according to (2) and (3), we have $\bigcup_{\alpha>0}(\alpha e + C) \subset \mathbb{R}^*_+\cdot \operatorname{icr} C + C \subset \operatorname{icr} C + \operatorname{vcl} C = \operatorname{icr} C$, hence $\bigcup_{\alpha>0}(\alpha e + C) \subset \operatorname{icr} C$. In order to prove the converse inclusion, let $x \in \operatorname{icr} C$. Then, for $y := -e \in \operatorname{span}(C - C)$, there exists $\lambda > 0$ such that $x + \lambda y \in C$, i.e., $x \in \lambda e + C$, which shows that $x \in \bigcup_{\alpha>0}(\alpha e + C)$. Hence the inclusion $\operatorname{icr} C \subset \bigcup_{\alpha>0}(\alpha e + C)$ also holds. Thus the first claimed equality is proven. Concerning the second one, firstly consider an arbitrary $\beta < 0$. Since $e \in \operatorname{icr} C$, by (2) and (3) it follows that $(-\beta)e + \operatorname{vcl} C \subset \mathbb{R}^*_+\cdot \operatorname{icr} C + \operatorname{vcl} C = \operatorname{icr} C$, hence $\operatorname{vcl} C \subset \beta e + \operatorname{icr} C$. Consequently, we get $\operatorname{vcl} C \subset \bigcap_{\beta<0}(\beta e + \operatorname{icr} C)$. In order to prove the converse inclusion, let $x \in \bigcap_{\beta<0}(\beta e + \operatorname{icr} C)$. Then, for every $n \in \mathbb{N}$, we have $x \in (-1/n)e + \operatorname{icr} C$, hence $x + \frac{1}{n}e \in C$. Since the sequence $\lambda_n := \frac{1}{n} > 0$ converges to 0, it follows that $x \in \operatorname{vcl} C$. We infer that $\bigcap_{\beta<0}(\beta e + \operatorname{int} C) \subset \operatorname{vcl} C$, which ends the proof. □

For any set-valued map $F: S \to 2^Y$, we denote by $\operatorname{dom} F := \{x \in S \mid F(x) \neq \emptyset\}$, $\operatorname{gr} F := \{(x, y) \in S \times Y \mid y \in F(x)\}$, and $\operatorname{epi}_C F := \{(x, y) \in S \times Y \mid y \in F(x) + C\}$, the effective domain, the graph, and the epigraph of F, respectively. For every set $A \subset Y$, we denote by $F^{-1}(A) := \{x \in S \mid F(x) \cap A \neq \emptyset\}$ the inverse image of A by F. A vector-valued function $f: S \to Y$ is said to be a selection of F if $f(x) \in F(x)$ for all $x \in S$ (in this case it is understood that $S = \operatorname{dom} F$).

Following Borwein (1977), we say that a set-valued map $F: S \to 2^Y$ is C-convex, if for all $x, x' \in S$ and $t \in [0, 1]$ we have

$$(1-t)F(x) + tF(x') \subset F((1-t)x + tx') + C,$$

which means that $\operatorname{epi}_C F$ is convex. Note that the effective domain of a C-convex set-valued map is convex. According to Kuroiwa (1996), the set-valued map $F: S \to 2^Y$ is called C-quasiconvex, if for all $x, x' \in S$ and $t \in [0, 1]$ we have

$$(F(x) + C) \cap (F(x') + C) \subset F((1-t)x + tx') + C,$$

which means that the lower level set $F^{-1}(y - C) := \{x \in S \mid F(x) \cap (y - C) \neq \emptyset\} = \{x \in S \mid y \in F(x) + C\}$ is convex, for each $y \in Y$. It is easily seen that F is C-quasiconvex whenever it is C-convex, since the cone C is convex. Note also that if C is solid then the effective domain of any C-quasiconvex set-valued map is convex, since in this case $Y = C - C$ (according to Lemma 1.13 in the monograph of Jahn, 1986), hence Y is directed, i.e., for all $x, y \in Y$ there exists $z \in Y$ such that $x \leq_C z$ and $y \leq_C z$.

A vector-valued function $f: S \to Y$ is said to be C-convex (resp., C-quasiconvex) if the set-valued map $F: S \to 2^Y$, defined for all $x \in S$ by $F(x) := \{f(x)\}$, is C-convex (resp., C-quasiconvex). Thus, f is C-convex if and only if

$$f((1-t)x + tx') \leq_C (1-t)f(x) + tf(x')$$

for all $x, x' \in S$ and $t \in [0, 1]$. Similarly, f is C-quasiconvex if and only if for all $x, x' \in S$, $t \in [0, 1]$ and $y \in Y$ we have

$$f((1-t)x + tx') \leq_C y \quad \text{whenever} \quad f(x) \leq_C y \quad \text{and} \quad f(x') \leq_C y.$$

Remark 2.1 Consider the particular case where $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$ and let $f = (f_1, \ldots, f_n): S \to \mathbb{R}^n$ be a vector function. It is easily seen that f is $\mathbb{R}^n_+$-convex if and only if it is componentwise convex. Similarly, as shown by Luc (1989), f is $\mathbb{R}^n_+$-quasiconvex if and only if it is componentwise quasiconvex. Note that this characterization of C-quasiconvexity has been extended by using extreme directions of the nonnegative polar of C by Luc (1989) and Benoist et al. (2003).

In what follows, it will be convenient to define, for every pair $(x, x') \in X \times X$, the function $\ell_{x,x'}: [0, 1] \to X$ by

$$\ell_{x,x'}(t) := (1-t)x + tx' \quad \text{for all } t \in [0, 1].$$

Our first result gives a characterization of cone-convex maps.

Proposition 2.1 *For any map $F: S \to 2^Y$ the following assertions are equivalent:*

1 *F is C-convex.*
2 *For all $x, x' \in S$ the composite map $F \circ \ell_{x,x'} : [0,1] \to 2^Y$ is C-convex.*
3 *For all (x, v), $(x', v') \in \operatorname{gr} F$ the set-valued map $F \circ \ell_{x,x'}$ admits a selection φ such that $\varphi(t) \leq_C (1-t)v + tv'$ for every $t \in [0,1]$.*

Proof The implication $1° \Longrightarrow 2°$ is obvious.

$2° \Longrightarrow 3°$. Suppose that 2° holds and let $(x, v), (x', v') \in \operatorname{gr} F$. Then we have $x, x' \in S$, $v \in F(x) = F \circ \ell_{x,x'}(0)$ and $v' \in F(x') = F \circ \ell_{x,x'}(1)$. Since $F \circ \ell_{x,x'}$ is C-convex, it follows that $(1-t)v + tv' \in (1-t)F \circ \ell_{x,x'}(0) + tF \circ \ell_{x,x'}(1) \subset F \circ \ell_{x,x'}(t) + C$ for all $t \in [0,1]$. Hence, for each $t \in [0,1]$ we can choose an element $v_t \in F \circ \ell_{x,x'}(t)$ such that $(1-t)v + tv' \in v_t + C$, i.e. $v_t \leq_C (1-t)v + tv'$. Then, the function $\varphi : [0,1] \to Y$, defined for all $t \in [0,1]$ by $\varphi := v_t$, is a selection of $F \circ \ell_{x,x'}$, which satisfies the property in demand.

$3° \Longrightarrow 1°$. Assume that 3° holds and let $x, x' \in S$ and $t \in [0,1]$. For any $y \in (1-t)F(x) + tF(x')$ there exist $v \in F(x)$ and $v' \in F(x')$ such that $y \in (1-t)v + tv'$. Since (x, v), $(x', v') \in \operatorname{gr} F$, we infer by assumption 3° the existence of a selection $\varphi : [0,1] \to Y$ of $F \circ \ell_{x,x'}$ with $\varphi(t) \leq_C (1-t)v + tv'$. It follows that $y \in (1-t)v + tv' \in \varphi(t) + C \subset F \circ \ell_{x,x'}(t) + C = F((1-t)x + tx') + C$. Hence $(1-t)F(x) + tF(x') \subset F((1-t)x + tx') + C$. Thus F is C-convex. □

The following result gives a characterization of cone-quasiconvex set-valued maps. We omit its proof since it is similar to that of Proposition 3.2.

Proposition 2.2 *For any set-valued map $F : S \to 2^Y$ the following assertions are equivalent:*

1 *F is C-quasiconvex.*
2 *For all $x, x' \in S$ the set-valued map $F \circ \ell_{x,x'}$ is C-quasiconvex.*
3 *For all (x, v), $(x', v') \in \operatorname{gr} F$ and $y \in Y$ with $v \leq_C y$, $v' \leq_C y$ the set-valued map $F \circ \ell_{x,x'}$ admits a selection φ such that $\varphi(0) \leq_C v$, $\varphi(1) \leq_C v'$ and $\varphi(t) \leq_C y$ for every $t \in]0,1[$.*

Corollary 2.1 *Let $F : S \to 2^Y$ be a set-valued map satisfying the property that for all $(x, v), (x', v') \in \operatorname{gr} F$ the composite set-valued map $F \circ \ell_{x,x'}$ admits a C-convex (resp., C-quasiconvex) selection φ such that $\varphi(0) \leq_C v$ and $\varphi(1) \leq_C v'$. Then F is C-convex (resp., C-quasiconvex).*

Proof It is easily seen that condition 3° in Proposition 2.1 (resp., Proposition 2.2) holds. The conclusion immediately follows. □

It is well-known that a function $f : S \to \mathbb{R}$ is quasiconvex if, and only if, for each $y \in \mathbb{R}$ the strict lower level set $\{x \in S \mid f(x) < y\}$ is convex. The following result shows that such a characterization also holds for set-valued maps, under the additional assumption that C is vectorially closed.

Theorem 2.1 *Let $F : S \to 2^Y$ be a set-valued map.*

(a) *If F is C-quasiconvex, then $F^{-1}(y - \operatorname{icr} C)$ is convex for all $y \in Y$.*
(b) *If C is vectorially closed and $F^{-1}(y - \operatorname{icr} C)$ is convex for all $y \in Y$, then the set-valued map F is C-quasiconvex.*

Proof (a) Assume that F is C-quasiconvex and let $y \in Y$. By Lemma 2.1, we have

$$F^{-1}(y - \operatorname{icr} C) = F^{-1}\left(y - \bigcup_{\alpha>0} (\alpha e + C)\right) = \bigcup_{\alpha>0} F^{-1}(y - \alpha e - C).$$

The set-valued map F being C-quasiconvex, the level sets of type $F^{-1}(z - C)$, $z \in Y$, are convex. In particular, $F^{-1}(y - \alpha e - C)$ is convex for every $\alpha > 0$. Taking into account that $F^{-1}(y - \alpha_1 e - C) \subset F^{-1}(y - \alpha_2 e - C)$ whenever $\alpha_1 \geq \alpha_2 > 0$, we can deduce that $\bigcup_{\alpha>0} F^{-1}(y - \alpha e - C)$, i.e. $F^{-1}(y - \operatorname{icr} C)$, is convex.

(b) Assume that $F^{-1}(y - \operatorname{icr} C)$ is convex for all $y \in Y$. Let $z \in Y$. Under the hypothesis that C is vectorially closed, i.e., $C = \operatorname{vcl} C$, it follows by Lemma 2.1 that

$$F^{-1}(z - C) = F^{-1}\left(z - \bigcap_{\beta<0} (\beta e + \operatorname{icr} C)\right) = \bigcap_{\beta<0} F^{-1}(z - \beta e - \operatorname{icr} C).$$

Since for every $\beta < 0$ the set $F^{-1}(z - \beta e - \operatorname{icr} C)$ is convex, it follows that the set $F^{-1}(z - C)$ is convex. □

Remark 2.2 As shown by the following example, the assumption on the vectorial closeness of C is essential in Theorem 2.1(b), even if F is single-valued.

Example 2.1 Let $S := [-1, 1] \subset X := \mathbb{R}$ and let $Y := \mathbb{R}^2$ be the two-dimensional Euclidean space, partially ordered by the lexicographical cone

$$C := (\mathbb{R}_+^* \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}_+).$$

This convex cone having a nonempty topological interior, $\operatorname{int} C = \mathbb{R}_+^* \times \mathbb{R}$, we have $\operatorname{icr} C = \operatorname{int} C$ and $\operatorname{vcl} C = \operatorname{cl} C$. Actually we have $\operatorname{vcl} C = \mathbb{R}_+ \times \mathbb{R} \neq C$, hence C is not vectorially closed. Consider the function $f : S \to Y$ defined by $f(x) = (0, 1 - x^2)$ for all $x \in S$. It is easily seen that $f^{-1}(y - \operatorname{icr} C)$ is convex for all $y \in Y$. However, f is not C-quasiconvex, since the level set $f^{-1}((0, 0) - C) = \{-1, 1\}$ is not convex.

3 Explicit cone-quasiconvexity

As in the previous section, in what follows Y will denote a real linear space, partially ordered by a relatively solid convex cone C.

Definition 3.1 A set-valued map $F : S \to 2^Y$ will be called explicitly C-quasiconvex if for all $x, x' \in S$ and $t \in]0, 1[$ the following inclusion holds:

$$(F(x) + C) \cap \left(F(x') + \operatorname{icr} C\right) \subset F((1 - t)x + tx') + \operatorname{icr} C.$$

A function $f : S \to Y$ will be called explicitly C-quasiconvex if the set-valued map $F : S \to 2^Y$ defined for all $x \in S$ by $F(x) = \{f(x)\}$ is explicitly C-quasiconvex.

Remark 3.1 It is easily seen that a vector-valued function $f : S \to Y$ is explicitly C-quasiconvex if and only if, for all $y \in Y$ and $x, x' \in S$ such that $f(x) \leq_C y$ and $f(x') <_C y$ and every $t \in]0, 1[$, we have $f((1 - t)x + tx') <_C y$. In particular, it follows that f possesses the following property:

$$f((1 - t)x + tx') <_C f(x) \quad \text{whenever } x, x' \in S,\ f(x') <_C f(x),\ t \in]0, 1[. \tag{4}$$

Note that property (4) characterizes semistrictly quasiconvex functions in the particular case when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$. In general, however, a vector-valued function $f : S \to Y$ which satisfies (4) is not necessarily explicitly C-quasiconvex, even if it is C-quasiconvex, as shown by the following example.

Example 3.1 Let $S := [-1, 1] \subset X := \mathbb{R}$ and let $Y := \mathbb{R}^2$ be the two-dimensional Euclidean space, partially ordered by the standard ordering cone $C := \mathbb{R}^2_+$. In this case we obviously have $\operatorname{int} C = \operatorname{icr} C = \mathbb{R}^*_+ \times \mathbb{R}^*_+$. Consider the function $f = (f_1, f_2) : S \to Y$ defined for all $x \in S$ by

$$f(x) = (\min\{0, -x\}, x).$$

Since both functions f_1 and f_2 are quasiconvex in the usual sense, the function f is C-quasiconvex according to Remark 2.1. Moreover, property (4) trivially holds, since there are no x, x' in S such that $f(x') <_C f(x)$. However, the function f is not explicitly C-quasiconvex, since $(f(-1) + C) \cap (f(1) + \operatorname{icr} C) \not\subset f(0) + \operatorname{icr} C$.

The following result shows that, by analogy to cone-quasiconvex maps, explicitly cone-quasiconvex maps can also be characterized in terms of (large and strict) lower level sets.

Proposition 3.1 *A set-valued map $F : S \to 2^Y$ is explicitly C-quasiconvex if and only if, for all $y \in Y$ and $(x, x') \in F^{-1}(y - C) \times F^{-1}(y - \operatorname{icr} C)$, we have*

$$(1 - t)x + tx' \in F^{-1}(y - \operatorname{icr} C) \quad \text{for every } t \in]0, 1[. \tag{5}$$

Proof Assume that F is explicitly C-quasiconvex. Consider some arbitrary $y \in Y$ and $(x, x') \in F^{-1}(y - C) \times F^{-1}(y - \operatorname{icr} C)$. Then $y \in (F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big)$. By explicit C-quasiconvexity of F it follows that $y \in F((1 - t)x + tx') + \operatorname{icr} C$ for all $t \in]0, 1[$, which means that (5) holds.

Conversely, suppose that (5) holds for all $y \in Y$ and $(x, x') \in F^{-1}(y - C) \times F^{-1}(y - \operatorname{icr} C)$. Let $x, x' \in S$ and $t \in]0, 1[$. For any $y \in (F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big)$ we have $(x, x') \in F^{-1}(y - C) \times F^{-1}(y - \operatorname{icr} C)$. By assumption (5) it follows that $(1 - t)x + tx' \in F^{-1}(y - \operatorname{icr} C)$, i.e. $y \in F((1 - t)x + tx') + \operatorname{icr} C$. Thus F is explicitly C-quasiconvex. □

The next result gives another characterization of explicitly C-quasiconvex maps, similar to Propositions 2.1 and 2.2.

Proposition 3.2 *For any set-valued map $F : S \to 2^Y$ the following assertions are equivalent:*

1° *F is explicitly C-quasiconvex.*
2° *For all $x, x' \in S$ the set-valued map $F \circ \ell_{x,x'}$ is explicitly C-quasiconvex.*
3° *For all (x, v), $(x', v') \in \operatorname{gr} F$ and $y \in Y$ with $v \leq_C y$, $v' <_C y$, the set-valued map $F \circ \ell_{x,x'}$ admits a selection φ such that $\varphi(0) \leq_C v$, $\varphi(1) \leq_C v'$ and $\varphi(t) <_C y$ for every $t \in]0, 1[$.*

Proof 1° $\Longrightarrow$ 2°. Assume that F is explicitly C-quasiconvex and let $x, x' \in S$. For all $t, t' \in [0, 1]$ and $s \in]0, 1[$ we have $\big(F \circ \ell_{x,x'}(t) + C\big) \cap \big(F \circ \ell_{x,x'}(t') + \operatorname{icr} C\big) = \big(F(\ell_{x,x'}(t)) + C\big) \cap \big(F(\ell_{x,x'}(t')) + \operatorname{icr} C\big) \subset F((1 - s)\ell_{x,x'}(t) + s\ell_{x,x'}(t')) + \operatorname{icr} C = F(\ell_{x,x'}((1 - s)t + st')) + \operatorname{icr} C = F \circ \ell_{x,x'}((1 - s)t + st') + \operatorname{icr} C$. Thus $F \circ \ell_{x,x'}$ is explicitly C-quasiconvex.

$2^\circ \implies 3^\circ$. Suppose that 2° holds. Let (x, v), $(x', v') \in \operatorname{gr} F$ and $y \in Y$ such that $v \leq_C y$, $v' <_C y$. Then $x, x' \in S$, $v \in F(x) = F \circ \ell_{x,x'}(0)$, $v' \in F(x') = F \circ \ell_{x,x'}(1)$ and $y \in (v + C) \cap (v' + \operatorname{icr} C)$, hence $y \in (F \circ \ell_{x,x'}(0) + C) \cap (F \circ \ell_{x,x'}(1) + \operatorname{icr} C)$. Since $F \circ \ell_{x,x'}$ is explicitly C-quasiconvex, it follows that $y \in F \circ \ell_{x,x'}(t) + \operatorname{icr} C$ for all $t \in]0, 1[$. Hence, for each $t \in]0, 1[$ we can choose an element $v_t \in F \circ \ell_{x,x'}(t)$ such that $y \in v_t + \operatorname{icr} C$. It is easily seen that a selection $\varphi : [0, 1] \to Y$ of $F \circ \ell_{x,x'}$ satisfying the desired properties in 3° can be defined by $\varphi(0) := v$, $\varphi(1) := v'$ and $\varphi(t) := v_t$ for all $t \in]0, 1[$.

$3^\circ \implies 1^\circ$. Assume that 3° holds and let $x, x' \in S$ and $t \in]0, 1[$. For any $y \in (F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big)$ there exist $v \in F(x)$ and $v' \in F(x')$ such that $y \in (v + C) \cap \big(v' + \operatorname{icr} C\big)$. Then we have (x, v), $(x', v') \in \operatorname{gr} F$, $v \leq_C y$, and $v' <_C y$. By assumption 3° we infer the existence of a selection $\varphi : [0, 1] \to Y$ of $F \circ \ell_{x,x'}$ such that $\varphi(t) <_C y$. It follows that $y \in \varphi(t) + \operatorname{icr} C \subset F \circ \ell_{x,x'}(t) + \operatorname{icr} C = F((1-t)x + tx') + \operatorname{icr} C$. Thus $(F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big) \subset F((1-t)x + tx') + \operatorname{icr} C$. Consequently, F is explicitly C-quasiconvex. □

Corollary 3.1 *Let $F : S \to 2^Y$ be a set-valued map satisfying the property that for all $(x, v), (x', v') \in \operatorname{gr} F$ the set-valued map $F \circ \ell_{x,x'}$ admits an explicitly C-quasiconvex selection φ such that $\varphi(0) \leq_C v$ and $\varphi(1) \leq_C v'$, then F is explicitly C-quasiconvex.*

Proof We just have to prove that condition 3° in Proposition 3.2 holds. Let (x, v), $(x', v') \in \operatorname{gr} F$ and $y \in Y$ with $v \leq_C y$, $v' <_C y$. By hypothesis, there exists an explicitly C-quasiconvex selection φ of $F \circ \ell_{x,x'}$ such that $\varphi(0) \leq_C v$ and $\varphi(1) \leq_C v'$. Then we have $\varphi(0) \leq_C y$ and $\varphi(1) <_C y$. By explicit C-quasiconvexity of φ it follows that $\varphi(t) <_C y$ for every $t \in]0, 1[$. Thus φ satisfies the desired property in 3°. □

The relationship between explicit cone-quasiconvexity, cone-convexity, and cone-quasiconvexity is emphasized by the following results.

Proposition 3.3 *Every C-convex set-valued map is explicitly C-quasiconvex.*

Proof Let $F : S \to 2^Y$ be a C-convex set-valued map. Let $x, x' \in S$ and $t \in]0, 1[$. By C-convexity of F and by recalling the properties (2) and (3), we infer that, for all $y \in (F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big)$,

$$\begin{aligned} y = (1-t)y + ty &\in (1-t)\,(F(x) + C) + t\,\big(F(x') + \operatorname{icr} C\big) \\ &\subset (1-t)F(x) + tF(x') + C + \operatorname{icr} C \\ &\subset F((1-t)x + tx') + \operatorname{icr} C. \end{aligned}$$

Hence $(F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big) \subset F((1-t)x + tx') + \operatorname{icr} C$. Thus F is explicitly C-quasiconvex. □

Proposition 3.4 *If C is vectorially closed, then every explicitly C-quasiconvex set-valued map is C-quasiconvex.*

Proof Let $F : S \to 2^Y$ be an explicitly C-quasiconvex set-valued map. Since C is vectorially closed, according to Theorem 2.1, we just have to prove that $F^{-1}(y - \operatorname{icr} C)$ is convex for all $y \in Y$. To this end, consider an arbitrary point $y \in Y$, and let $x, x' \in F^{-1}(y - \operatorname{icr} C)$ and $t \in]0, 1[$. By explicit C-quasiconvexity of F we have

$$\begin{aligned} y &\in (F(x) + \operatorname{icr} C) \cap \big(F(x') + \operatorname{icr} C\big) \\ &\subset (F(x) + C) \cap \big(F(x') + \operatorname{icr} C\big) \\ &\subset F((1-t)x + tx') + \operatorname{icr} C, \end{aligned}$$

hence $(1-t)x + tx' \in F^{-1}(y - \operatorname{icr} C)$. Thus $F^{-1}(y - \operatorname{icr} C)$ is convex. □

Remark 3.2 The assumption on the closeness of the ordering cone C is essential in Proposition 3.4, even if F is single-valued. Indeed, consider for instance the function f defined in Example 2.1. We have already seen that it is not C-quasiconvex. However, f is explicitly C-quasiconvex, since for all $x, x' \in S$ and $t \in]0,1[$ we actually have $(f(x)+C) \cap \big(f(x')+\operatorname{icr} C\big) = f((1-t)x+tx')+\operatorname{icr} C$.

The following auxiliary result will be useful in proof of Proposition 3.5 below.

Lemma 3.1 *Let A and B be two subsets of Y such that $A+C=B+C$. Then*

$$A+\operatorname{icr} C = B+\operatorname{icr} C.$$

Proof By using (3) it can be easily shown that $C+\operatorname{icr} C=\operatorname{icr} C$. Since $A+C=B+C$, we infer $A+\operatorname{icr} C = A+C+\operatorname{icr} C = B+C+\operatorname{icr} C = B+\operatorname{icr} C$. □

Proposition 3.5 *Let $F\colon S \to 2^Y$ and $G\colon S \to 2^Y$ be two set-valued maps satisfying the property that $F(x)+C=G(x)+C$ for all $x \in S$. Then, F is C-quasiconvex (explicitly C-quasiconvex, C-convex) if and only if G is C-quasiconvex (explicitly C-quasiconvex, C-convex).*

Proof For all $x \in S$ we have $F(x)+C=G(x)+C$. By Lemma 3.1 it follows that we also have $F(x)+\operatorname{icr} C = G(x)+\operatorname{icr} C$, for all $x \in S$. By definition of (explicit) C-quasiconvexity we infer that F is (explicitly) C-quasiconvex if and only if G is (explicitly) C-quasiconvex. On the other hand, we have $\operatorname{epi}_C(F) = \{(x,y) \in S \times Y \mid y \in F(x)+C\} = \{(x,y) \in S \times Y \mid y \in G(x)+C\} = \operatorname{epi}_C(G)$, which shows that F is C-convex if and only if G is C-convex. □

Remark 3.3 Following Rubinov (2000) we say that a subset A of Y is upward (with respect to C) if $A=A+C$, which actually means that the free disposal property in the sense of Debreu (1959) holds. By Proposition 3.5 it follows that the C-quasiconvexity (explicit C-quasiconvexity, C-convexity, respectively) of a set-valued map $F\colon S \to 2^Y$ reduces to the C-quasiconvexity (explicit C-quasiconvexity, C-convexity, respectively) of a set-valued map with upward values, namely $F+C$, defined for all $x \in S$ by $(F+C)(x) := F(x)+C$.

Corollary 3.2 *Let $F\colon S \to 2^Y$ be a set-valued map which admits a selection f such that, for each $x \in S$, $f(x)$ is a smallest element of $F(x)$. Then, F is C-quasiconvex (explicitly C-quasiconvex, resp., C-convex) if and only if function f is C-quasiconvex (explicitly C-quasiconvex, resp., C-convex).*

Proof For all $x \in S$ we have $f(x)+C \subset F(x)+C \subset f(x)+C+C = f(x)+C$, hence $f(x)+C=F(x)+C$. The conclusion directly follows by Proposition 3.5. □

We end this Section by showing that the notion of explicitly cone-quasiconvexity is a natural extension of the classical notion of explicit quasiconvexity.

Theorem 3.1 *A function $f: S \to \mathbb{R}^n$ is explicitly $\mathbb{R}^n_+$-quasiconvex if and only if it is componentwise explicitly quasiconvex.*

Proof The conclusion being obvious for $n=1$, we may suppose that $n \geq 2$. Assume that $f=(f_1,\ldots,f_n)$ is explicitly $\mathbb{R}^n_+$-quasiconvex. By Proposition 3.4, it follows that f is $\mathbb{R}^n_+$-quasiconvex, which means, according to Remark 2.1, that all functions $f_1,\ldots,f_n$ are quasiconvex in the usual sense. Suppose to the contrary that f_i is not semistrictly quasiconvex for some $i \in \{1, \ldots, n\}$. Since f_i is quasiconvex, we infer the existence of $x, x' \in S$ and $t \in]0, 1[$ satisfying the following condition:

$$f_i(x') < f_i((1-t)x + tx') = f_i(x). \tag{6}$$

Since f is explicitly $\mathbb{R}^n_+$-quasiconvex, it satisfies property (4). Consider the point $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, where $y_i := f_i(x)$ and $y_j := \max\{f_j(x), f_j(x')\} + 1$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. Then $f(x) \leq y$ and $f(x') < y$, in view of (6). Since f is explicitly $\mathbb{R}^n_+$-quasiconvex, we can deduce that $f((1-t)x + tx') < y$, which yields $f_i((1-t)x + tx') < y_i = f_i(x)$, contradicting (6).

Conversely suppose that $f_1, \ldots, f_n$ are explicitly quasiconvex in the usual sense. Let $y \in \mathbb{R}$, let $x, x' \in S$ be such that $f(x) \leq y$ and $f(x') < y$, and let $t \in]0, 1[$. For any $i \in \{1, \ldots, n\}$ we distinguish two possible situations: if $f_i(x) = f_i(x')$, then $f_i((1-t)x+tx') \leq \max\{f_i(x), f_i(x')\} = f_i(x') < y_i$, by quasiconvexity of f_i; if $f_i(x) \neq f_i(x')$, then $f_i((1-t)x+tx') < \max\{f_i(x), f_i(x')\} \leq y_i$, by semistrict quasiconvexity of f_i. Hence, in both cases, we have $f_i((1-t)x + tx') < y_i$. Thus $f((1-t)x + tx') < y$. In view of Remark 3.1, f is explicitly $\mathbb{R}^n_+$-quasiconvex. □

4 Set-valued optimization problems

Throughout this Section, we will restrict our attention to the particular case where $Y = \mathbb{R}^n$ is the n-dimensional Euclidean space with $n \geq 2$, partially ordered by the standard ordering cone $C = \mathbb{R}^n_+$. As usual, the coordinates of any vector $y \in \mathbb{R}^n$ with respect to the canonical basis $\{e^1, \ldots, e^n\}$ of $\mathbb{R}^n$ will be denoted by $y_1, \ldots, y_n$. For any subset Z of $\mathbb{R}^n$ we denote by

$$\operatorname{Min} Z := \{z \in Z \mid Z \cap (z - \mathbb{R}^n_+) = \{z\}\} = \{z \in Z \mid \nexists\, z' \in Z : z' \leq z \neq z'\},$$
$$\operatorname{WMin} Z := \{z \in Z \mid Z \cap (z - \operatorname{int} \mathbb{R}^n_+) = \emptyset\} = \{z \in Z \mid \nexists\, z' \in Z : z' < z\}$$

the sets of efficient points, and weakly efficient points of Z, respectively. According to Luc (1989), given a set-valued map $F \colon S \to 2^{\mathbb{R}^n}$, the efficient solutions and the weakly efficient solutions of the set-valued optimization problem

$$\text{(SVOP)} \quad \begin{cases} \text{Minimize} & F(x) \quad \text{w.r.t.} \quad \mathbb{R}^n_+ \\ \text{subject to} & x \in S \end{cases}$$

are defined as the elements of the following sets, respectively:

$$\operatorname{Eff}(S \mid F) := F^{-1}(\operatorname{Min} F(S)) \quad \text{and} \quad \operatorname{WEff}(S \mid F) := F^{-1}(\operatorname{WMin} F(S)).$$

Note that classical multicriteria optimization problems can be treated in the same framework, by considering objective maps with singleton values. The principal aim of this Section is to extend some recent results of Benoist (2003) and Popovici (2005) concerning the structure of efficient sets from explicitly quasiconvexity multicriteria optimization to explicitly quasiconvex set-valued optimization. To this end we firstly recall some basic definitions. A subset Z of $\mathbb{R}^n$ is called:

- K-radiant, where K is a cone of $\mathbb{R}^n$, if

$$\operatorname{ray}(z, z') := z + \mathbb{R}_+(z' - z) \subset Z \quad \text{for all} \quad z, z' \in Z,\ z \leq_K z';$$

- simply shaded, if Z is closed, upward (i.e., $Z = Z + \mathbb{R}^n_+$), and its topological boundary (which actually coincides with $\operatorname{WMin} Z$) is $\mathbb{R}^n_+$-radiant.

Remark 4.1 Every upward subset of $\mathbb{R}^n$ is $\mathbb{R}^n_+$-radiant, but its weakly efficient frontier is not necessarily $\mathbb{R}^n_+$-radiant. For instance, the set $Z := \{e^1, \dots, e^n\} + \mathbb{R}^n_+$ is upward, but WMin Z is not $\mathbb{R}^n_+$-radiant. Indeed, for $z := e^1$ and $z' := e^1 + \cdots + e^n$, we have $z, z' \in \text{WMin}\, Z$ and $z \leq z'$, but $\text{ray}(z, z') \not\subset \text{WMin}\, Z$.

Remark 4.2 As shown by Popovici (2005), if Z is an upward subset of $\mathbb{R}^n$, then the set WMin Z is $\mathbb{R}^n_+$-radiant if and only if it is $\mathbb{R}_+\{e^1, \dots, e^n\}$-radiant.

We are now ready to state our main results concerning optimization problems of type (SVOP) with explicitly quasiconvex objective set-valued maps. For notational convenience we set $I_n := \{1, \dots, n\}$.

Theorem 4.1 *Let $F: S \to 2^{\mathbb{R}^n}$ be a set-valued map. Assume that for all (x, v), $(x', v') \in \text{gr}\, F$ and $y \in \mathbb{R}^n$ with $v \leq y$, $v' < y$, the set-valued map $F \circ \ell_{x,x'}$ admits a componentwise upper semicontinuous selection φ such that $\varphi(0) \leq v$, $\varphi(1) \leq v'$ and $\varphi(t) < y$ for every $t \in]0, 1[$. Then* $\text{WMin}\,(F(S) + \mathbb{R}^n_+)$ *is $\mathbb{R}^n_+$-radiant.*

Proof Suppose to the contrary that $\text{WMin}\,(F(S) + \mathbb{R}^n_+)$ is not $\mathbb{R}^n_+$-radiant. In view of Remark 4.2 and taking into account that $F(S) + \mathbb{R}^n_+$ is upward, we infer the existence of some $i \in I_n$ for which $\text{WMin}\,(F(S) + \mathbb{R}^n_+)$ is not $\mathbb{R}_+ e^i$-radiant. Consequently, we can find some z, $z' \in \text{WMin}\,(F(S) + \mathbb{R}^n_+)$ and $\alpha \in \mathbb{R}_+$ such that $z' \in z + \mathbb{R}_+ e^i$ and

$$y := z + \alpha(z' - z) \notin \text{WMin}\,(F(S) + \mathbb{R}^n_+).$$

Since z', $y \in z + \mathbb{R}_+ e^i$, we have $z_j = z'_j = y_j$ for all $j \in I_n \setminus \{i\}$. It is easily seen that we also have $z_i < z'_i < y_i$, since z, $z' \in \text{WMin}\,(F(S) + \mathbb{R}^n_+)$ and $y \notin \text{WMin}\,(F(S) + \mathbb{R}^n_+)$. Given that $z \in F(S) + \mathbb{R}^n_+$, there exists $v \in F(S)$ such that $v \leq z$. On the other hand, since $y \in z + \mathbb{R}_+ e^i \subset F(S) + \mathbb{R}^n_+$ and $y \notin \text{WMin}\,(F(S) + \mathbb{R}^n_+)$, there exist $v' \in F(S)$ and $c \in \mathbb{R}^n_+$ such that $v' + c < y$. Let $x, x' \in S$ be such that $v \in F(x)$ and $v' \in F(x')$, i.e. (x, v), $(x', v') \in \text{gr}\, F$. Since $v \leq y$ and $v' < y$, we infer the existence of a componentwise upper semicontinuous function $\varphi = (\varphi_1, \dots, \varphi_n)\colon [0, 1] \to \mathbb{R}^n$ such that $\varphi(t) \in F \circ \ell_{x,x'}(t)$ for all $t \in [0, 1]$, $\varphi(0) \leq v$, $\varphi(1) \leq v'$, and

$$\varphi(t) < y \quad \text{for all } t \in]0, 1[. \tag{7}$$

Recalling that $\varphi(0) \leq v \leq z$ and $z_i < z'_i$, we have that $\varphi_i(0) < z'_i$. Thus, by upper semicontinuity of φ_i, we can deduce that $\varphi_i(s) < z'_i$ for a small enough $s \in]0, 1[$. By (7) and recalling that $y_j = z'_j$ for all $j \in I_n \setminus \{i\}$, we infer that $\varphi_j(s) < z'_j$ for all $j \in I_n \setminus \{i\}$. Hence $\varphi(s) < z'$. Since $\varphi(s) \in F \circ \ell_{x,x'}(s) \in F(S) \subset F(S) + \mathbb{R}^n_+$, it follows that $z' \notin \text{WMin}\,(F(S) + \mathbb{R}^n_+)$, contradicting the initial choice of z'. □

Corollary 4.1 *Let $F : S \to 2^{\mathbb{R}^n}$ be a set-valued map satisfying the property that for all $(x, v), (x', v') \in \text{gr}\, F$ the set-valued map $F \circ \ell_{x,x'}$ admits a componentwise upper semicontinuous explicitly quasiconvex selection φ such that $\varphi(0) \leq v$ and $\varphi(1) \leq v'$. Then* $\text{WMin}\,(F(S) + \mathbb{R}^n_+)$ *is $\mathbb{R}^n_+$-radiant.*

Proof The conclusion directly follows by Theorem 4.1, by an argument similar to that used in the proof of Corollary 3.1. □

Remark 4.3 Under the hypotheses of Theorem 4.1 (resp., Corollary 4.1), F is explicitly $\mathbb{R}^n_+$-quasiconvex, according to Proposition 3.2 (resp., Corollary 3.1).

Corollary 4.2 *Let $F\colon S \to 2^{\mathbb{R}^n}$ be an explicitly $\mathbb{R}^n_+$-quasiconvex set-valued map such that a smallest element exists in each of its values. Assume that F is lower semicontinuous along line segments, i.e., for all $x, x' \in S$ the set-valued map $F \circ \ell_{x,x'}$ is lower semicontinuous. Then* WMin $(F(S) + \mathbb{R}^n_+)$ *is $\mathbb{R}^n_+$-radiant.*

Proof Let $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ be the function which assigns to each value $F(x)$ of F its smallest element $f(x)$. Let $(x, v), (x', v') \in \operatorname{gr} F$ and consider the function $\varphi = (\varphi_1, \ldots, \varphi_n)\colon [0,1] \to \mathbb{R}^n$, defined for all $t \in [0,1]$ by

$$\varphi(t) := f \circ \ell_{x,x'}(t).$$

Obviously φ is a selection of the set-valued map $F \circ \ell_{x,x'}$ with $\varphi(0) = f(x) \leq v$ and $\varphi(1) = f(x') \leq v'$. The set-valued map F being explicitly $\mathbb{R}^n_+$-quasiconvex, it follows by Corollary 3.2 that f is explicitly $\mathbb{R}^n_+$-quasiconvex. Then $f \circ \ell_{x,x'}$, i.e. φ, is explicitly $\mathbb{R}^n_+$-quasiconvex, according to Proposition 3.2. Thus, in view of Corollary 4.1, in order to prove that WMin $(F(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant, we just have to show that for each $i \in I_n$ the function φ_i is upper semicontinuous. To this end, by means of the projection function $p_i\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined for all $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$ by $p_i(t, v) := v_i$, we associate to $F \circ \ell_{x,x'}$ the marginal function $\mu\colon [0,1] \to \mathbb{R}$ given by

$$\mu(t) := \inf\{p_i(t, v) \mid v \in F \circ \ell_{x,x'}(t)\} \quad \text{for all } t \in [0,1].$$

Since for every $t \in [0,1]$ the point $f((1-t)x + tx')$ is a smallest element of the set $F \circ \ell_{x,x'}(t) = F((1-t)x + tx')$, it is easily seen that $\mu(t) = f_i((1-t)x + tx') = f_i \circ \ell_{x,x'}(t)$. Hence $\mu = f_i \circ \ell_{x,x'}$. Finally, recalling that the set-valued map $F \circ \ell_{x,x'}$ is lower semicontinuous and taking into account that p_i is continuous, we infer by the well-known Maximum Theorem (see, e.g., Theorem 1.4.16 in the monograph of Aubin and Frankowska, 1990) that the marginal function μ, i.e. $f_i \circ \ell_{x,x'}$, is upper semicontinuous. □

We end our paper by presenting the announced applications of Theorem 4.1 to the study of the Pareto reducibility of problem (SVOP) and the contractibility of its efficient outcome set.

For every nonempty subset I of I_n, consider the convex cone

$$C_I := \{y \in \mathbb{R}^n \mid y_i \geq 0, \forall\, i \in I\}.$$

According to Luc (1989), the set of efficient points of any subset Z of $\mathbb{R}^n$ with respect to C_I is defined by

$$\begin{aligned} \operatorname{Min}_I Z &:= \{z \in Z \mid Z \cap (z - C_I) \subset z + C_I\} \\ &= \{z \in Z \mid \forall\, z' \in Z : (z'_i \leq z_i, \forall\, i \in I) \Rightarrow (z'_i = z_i, \forall\, i \in I)\}. \end{aligned}$$

Then, the set $\operatorname{Eff}_I(S \mid F) := F^{-1}(\operatorname{Min}_I F(S))$ represents the set of efficient solutions of the following set-valued optimization problem associated to (SVOP):

$$(\text{SVOP}_I) \quad \begin{cases} \text{Minimize} & F(x) \quad \text{w.r.t.} \quad C_I \\ \text{subject to} & x \in S. \end{cases}$$

Obviously, (SVOP_{I_n}) actually means (SVOP), hence $\operatorname{Eff}_{I_n}(S \mid F) = \operatorname{Eff}(S \mid F)$. Following Popovici (2005), we will say that problem (SVOP) is Pareto reducible if the set of its weakly efficient solutions can be represented as the union of the efficient solutions of all associated problems of type (SVOP_I), i.e.

$$\text{WEff}(S \mid F) = \bigcup_{\emptyset \neq I \subset I_n} \text{Eff}_I(S \mid F).$$

Proposition 4.1 *Under the hypotheses of Theorem 4.1 the problem* (SVOP) *is Pareto reducible.*

Proof According to the main theorem of Popovici (2005), the weakly efficient frontier WMin Z of any upward set $Z \subset \mathbb{R}^n$ is $\mathbb{R}^n_+$-radiant if and only if

$$\text{WMin}\, Z = \bigcup_{\emptyset \neq I \subset I_n} \text{Min}_I\, Z. \tag{8}$$

Obviously, the set $F(S) + \mathbb{R}^n_+$ is upward. Moreover, the set WMin $(F(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant, by virtue of Theorem 4.1. We infer by (8) applied for $Z = F(S) + \mathbb{R}^n_+$ that $\text{WMin}\,(F(S) + \mathbb{R}^n_+) = \bigcup_{\emptyset \neq I \subset I_n} \text{Min}_I\,(F(S) + \mathbb{R}^n_+)$. Taking into account that $\text{Min}_I F(S) = F(S) \cap \text{Min}_I(F(S) + \mathbb{R}^n_+)$ for each nonempty $I \subset I_n$, and $\text{WMin}\, F(S) = F(S) \cap \text{WMin}\,(F(S) + \mathbb{R}^n_+)$, it follows that

$$\begin{aligned}
\text{WMin}\, F(S) &= F(S) \cap \text{WMin}\,(F(S) + \mathbb{R}^n_+) \\
&= F(S) \cap \left(\bigcup_{\emptyset \neq I \subset I_n} \text{Min}_I\,(F(S) + \mathbb{R}^n_+) \right) \\
&= \bigcup_{\emptyset \neq I \subset I_n} (F(S) \cap \text{Min}_I\,(F(S) + \mathbb{R}^n_+)) \\
&= \bigcup_{\emptyset \neq I \subset I_n} \text{Min}_I\, F(S).
\end{aligned}$$

Hence we also have

$$\begin{aligned}
\text{WEff}\, S &= F^{-1}(\text{WMin}\, F(S)) \\
&= F^{-1}\left(\bigcup_{\emptyset \neq I \subset I_n} \text{Min}_I\, F(S) \right) \\
&= \bigcup_{\emptyset \neq I \subset I_n} F^{-1}(\text{Min}_I\, F(S)) \\
&= \bigcup_{\emptyset \neq I \subset I_n} \text{Eff}_I S.
\end{aligned}$$

Thus problem (SVOP) is Pareto reducible. □

Let us now turn our attention on the contractibility of efficient sets. Recall that a subset A of $\mathbb{R}^n$ is said to be strongly contractible if there exists a point $a \in A$ and a continuous function h_A: $[0, 1] \times A \to A$ such that $h_A(0, y) = y$ and $h_A(1, y) = a = h_A(t, a)$ for all $y \in A$ and $t \in [0, 1]$. It is well known that every (strongly) contractible set is arcwise connected, hence connected. A well-known theorem of Peleg (1972) states that the efficient frontier of any closed convex nonempty set in $\mathbb{R}^n$ having compact sections with respect to $\mathbb{R}^n_+$ is contractible. After the publication of this result the literature devoted to the connectedness of efficient sets in both convex and nonconvex vector optimization problems has quickly grown. However, only a few results have been obtained in what concerns arcwise connectedness or contractibility of efficient sets in nonconvex vector optimization. A valuable extension of the Peleg's Theorem has been recently obtained by Benoist (2003), which asserts that the efficient frontier of any nonempty simply shaded set in $\mathbb{R}^n$ having compact sections with respect to $\mathbb{R}^n_+$ is strong contractible. We end our paper by showing how this result can be used in explicitly quasiconvex set-valued optimization.

The following result extends Lemma 3.3 in the paper of Huy and Yen (2005).

Proposition 4.2 *Assume that S is a nonempty subset of a real linear topological Hausdorff space X and let $F\colon S \to 2^{\mathbb{R}^n}$ be an upper semicontinuous set-valued map with nonempty compact values, such that for every $v \in F(S)+\mathbb{R}^n_+$ the lower level set $F^{-1}(v-\mathbb{R}^n_+)$ is compact. Then the following assertions hold:*

1° *The set $F(S)+\mathbb{R}^n_+$ is closed.*
2° *For every $v \in \mathbb{R}^n$, the lower section $\Omega(v) := (F(S)+\mathbb{R}^n_+) \cap (v-\mathbb{R}^n_+)$ is compact.*

Proof 1° Let $(z^k)_{k\in\mathbb{N}}$ be a sequence of points in $F(S)+\mathbb{R}^n_+$, which converges to a point $z \in \mathbb{R}^n$. We have to prove that $z \in F(S)+\mathbb{R}^n_+$.

For each $k \in \mathbb{N}$ there exist $x^k \in S$, $y^k \in F(x^k)$ and $c^k \in \mathbb{R}^n_+$ such that $z^k = y^k + c^k$. The sequence $(z^k)_{k\in\mathbb{N}}$ being convergent, it is bounded, hence there exists $v \in \mathbb{R}^n$ such that $z^k \in v-\mathbb{R}^n_+$ for all $k \in \mathbb{N}$. It follows that, for every $k \in \mathbb{N}$ we have $v \in z^k+\mathbb{R}^n_+ = y^k+c^k+\mathbb{R}^n_+ \subset F(x^k)+\mathbb{R}^n_++\mathbb{R}^n_+ = F(x^k)+\mathbb{R}^n_+$, hence $x^k \in F^{-1}(v-\mathbb{R}^n_+)$ and $y^k \in F(x^k) \subset F(F^{-1}(v-\mathbb{R}^n_+))$. By hypothesis, the level set $F^{-1}(v-\mathbb{R}^n_+)$ is compact, since $v \in F(S)+\mathbb{R}^n_+$ (indeed, $v \in z^1+\mathbb{R}^n_+ \subset F(S)+\mathbb{R}^n_++\mathbb{R}^n_+ = F(S)+\mathbb{R}^n_+$). By a classical argument in Set-Valued Analysis (see, e.g., Theorem 2.1 in the early paper of Hiriart-Urruty (1985) or Proposition 2.5.8 in the monograph of Göpfert et al. (2003), it follows that $F(F^{-1}(v-\mathbb{R}^n_+))$ is compact, as the image of a compact set under an upper semicontinuous map with nonempty compact values. Thus, passing to a subsequence if necessary, we can assume without loss of generality that $(y^k)_{k\in\mathbb{N}}$ converges to a point $y \in F(F^{-1}(v-\mathbb{R}^n_+))$. Recalling that $(z^k)_{k\in\mathbb{N}}$ converges to z, and $z^k = y^k + c^k$ for all $k \in \mathbb{N}$, we infer that $(c^k)_{k\in\mathbb{N}}$ converges to $z-y$. Since $c^k \in \mathbb{R}^n_+$ for all $k \in \mathbb{N}$, it follows that $z-y \in \operatorname{cl}\mathbb{R}^n_+ = \mathbb{R}^n_+$, hence $z \in y+\mathbb{R}^n_+ \subset F(F^{-1}(v-\mathbb{R}^n_+))+\mathbb{R}^n_+ \subset F(S)+\mathbb{R}^n_+$.

2° Let $v \in \mathbb{R}^n$. Since $\Omega(v)=\emptyset$ (hence compact) if $v \notin F(S)+\mathbb{R}^n_+$, we can assume in what follows that $v \in F(S)+\mathbb{R}^n_+$. Since $v-\mathbb{R}^n_+$ is closed, it follows by 1° that $\Omega(v)$ is also closed. Obviously, $\Omega(v)$ is upper bounded by v, i.e. $\Omega(v) \subset v-\mathbb{R}^n_+$. Thus, in order to prove that $\Omega(v)$ is compact, we just have to show that $\Omega(v)$ is lower bounded.

By hypothesis, $F^{-1}(v-\mathbb{R}^n_+)$ is compact, hence $F(F^{-1}(v-\mathbb{R}^n_+))$ is also compact, as the image of a compact set under an upper semicontinuous map with nonempty compact values. Since compact sets in $\mathbb{R}^n$ are lower bounded, we infer the existence of $w \in \mathbb{R}^n$ such that $F(F^{-1}(v-\mathbb{R}^n_+)) \subset w+\mathbb{R}^n_+$. We will end the proof by showing that $\Omega(v) \subset w+\mathbb{R}^n_+$. Indeed, let $u \in \Omega(v)$. By definition of $\Omega(v)$, there exist $y \in F(S)$ and $c, c' \in \mathbb{R}^n_+$ such that $u = y+c = v-c'$, i.e. $u-c = y = v-c-c'$. Taking into account that $y \in F(S)$ and $v-c-c' \in v-\mathbb{R}^n_+-\mathbb{R}^n_+ = v-\mathbb{R}^n_+$, it follows that $u-c \in F(S) \cap (v-\mathbb{R}^n_+)$, which means that $u-c \in F(x) \cap (v-\mathbb{R}^n_+)$ for some $x \in S$. Thus $F(x) \cap (v-\mathbb{R}^n_+) \neq \emptyset$ and, consequently, $x \in F^{-1}(v-\mathbb{R}^n_+)$. We deduce that $u-c \in F(x) \subset F(F^{-1}(v-\mathbb{R}^n_+)) \subset w+\mathbb{R}^n_+$, hence $u \in w+c+\mathbb{R}^n_+ \subset w+\mathbb{R}^n_++\mathbb{R}^n_+ = w+\mathbb{R}^n_+$. Since u was arbitrarily chosen in $\Omega(v)$, we infer that $\Omega(v) \subset w+\mathbb{R}^n_+$. □

Corollary 4.3 *Let S be a nonempty convex subset of a real linear topological Hausdorff space X and let $F: S \to 2^{\mathbb{R}^n}$ be an upper semicontinuous set-valued map with nonempty compact values, such that for every $v \in F(S) + \mathbb{R}^n_+$ the lower level set $F^{-1}(v - \mathbb{R}^n_+)$ is compact. Assume, in addition, that F satisfies the hypotheses of Theorem 4.1. Then* Min $F(S)$ *is strongly contractible.*

Proof Obviously $F(S) + \mathbb{R}^n_+$ is a an upward set. Actually, it is simply shaded, since it is closed and WMin $(F(S) + \mathbb{R}^n_+)$ is $\mathbb{R}^n_+$-radiant, as shown by Proposition 4.2(1°) and Theorem 4.1. Moreover, according to Proposition 4.2(2°), the lower sections of $F(S) + \mathbb{R}^n_+$ are compact. By Theorem 4.2 in the paper of Benoist (2003) it follows that Min $(F(S) + \mathbb{R}^n_+)$, i.e., Min $F(S)$, is strongly contractible. □

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